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TWO APPLICATIONS OF CHAIN SEQUENCES TO UNIVALENCE

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ABSTRACT

By a reformulation of some classical results on chain sequences, necessary and sufficient conditions are obtained for each function $F(z) = z/1 + \underline{a_1 z^{\alpha}/}1 + \underline{a_2 z^{\alpha}/}1 + \dots$, where $|a_n| \leq k_n$, $n = 1, 2, \dots$ and $\{k_n\}$ is a fixed chain sequence, to be univalent in the unit disk. This extends results of Thale, Perron (cf., O. Perron, Die Lehre von den Kettenbrüchen, vol. 2, Stuttgart, 1957, p. 148), and others. In addition, estimates of the starlike radius of the functions $F(z)$ are found. The second application relates chain sequences to the radii of univalence and of starlikeness of a class of functions $f(z)$ where the ratio $zf'(z)/f(z)$ has a certain type of C-fraction expansion. As an illustration of the result, the question of univalence and starlikeness of suitably normalized Bessel functions is considered.

TWO APPLICATIONS OF CHAIN SEQUENCES TO UNIVALENCE

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1. Introduction. A sequence of real numbers $k = \{k_n\}_{n=1}^{\infty}$ for which there exist g_{n-1}, $0 \leq g_{n-1} \leq 1$, such that $k_n = g_n(1 - g_{n-1})$ for $n = 1, 2, \ldots$ is called a chain sequence and the numbers g_{n-1} are the parameters of the sequence. In general, a chain sequence does not uniquely determine its parameters. However, Wall [15, p. 80] proves the existence of minimal and maximal parameter sequences, $\{m_n\}_{n=0}^{\infty}$ and $\{M_n\}_{n=0}^{\infty}$ respectively, such that $m_n \leq g_n \leq M_n$, $n = 0, 1, 2, \ldots$, for every parameter sequence $\{g_n\}_{n=0}^{\infty}$ of k. Throughout this paper, the maximal parameter sequence is a judicious choice, although not a necessary one unless so stated, in the application of the results.

The concept of a chain sequence was initiated by Pringsheim and Van Vleck in connection with the question of convergence of certain continued fractions. In §2 of this paper, these and subsequent results on chain sequences have been reformulated in a manner suitable for the particular applications presented here. The first of these applications extends some

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recent results on univalence of Scott and Merkes [7] to a wider class of functions. In addition, earlier theorems on the starlikeness of these functions are extended and, in some cases, improved. The second application is devoted to the univalence and starlikeness of a class of functions Π_f which is defined from the C-fraction expansion of the ratio $zf'(z)/f(z)$. The results obtained provide a simple numerical and theoretical method to estimate the radii of univalence and of starlikeness of the class. An application is made in §7 to the Bessel functions and some recent results on this topic [1, 2, 3] are extended and improved. Moreover, it is easy to apply the method to certain function allied to the hypergeometric functions by utilizing the continued fraction of Gauss.

2. A class of bounded continued fractions.

Let $\lambda = \{\lambda_n\}_{n=1}^{\infty}$ be a sequence of complex numbers such that $\lambda_n = 0$ implies $\lambda_{n+p} = 0$ for $p = 1, 2, \ldots$. The continued fraction

$$(2.1) \qquad \frac{1}{1} + \frac{\lambda_1}{1} + \frac{\lambda_2}{1} + \cdots + \frac{\lambda_n}{1} + \cdots ,$$

associated with λ, is said to be in the class B provided there is a real number $R \geq 1$ such that, for $n = 1, 2, \ldots$, the values of the terminating continued fraction

$$(2.2) \qquad \frac{1}{1} + \frac{\lambda'_1}{1} + \frac{\lambda'_2}{1} + \cdots + \frac{\lambda'_n}{1}$$

are in the disk $|w| \leq R$ whenever $|\lambda'_j| \leq |\lambda_j|$, $j = 1, 2, \ldots$. This definition relates the sequence λ to certain chain sequences. First, it is convenient to have the following elementary result.

LEMMA 2.1 . *Let A and $g \leq 1$ be positive numbers. For all λ in the disk $|\lambda| \leq A$, the values of s from the transformation $s = 1/(1-\lambda t)$ lie in the disk $|s| \leq 1/g$ if and only if the values of t are in $|t| \leq (1-g)/A$.*

Proof. Clearly $|s| \leq 1/g$ if and only if $|1-\lambda t| \geq g$. Since $|1-\lambda t| \geq 1-|t|\,|\lambda| \geq 1-A|t|$, the sufficiency follows. If $t = \rho e^{i\phi}$, where $\rho > (1-g)/A$, let $\lambda = re^{-i\phi}$ where $(1-g)/\rho < r \leq A$. Then $1-\lambda t = 1-r\rho < g$ so that $|s| > 1/g$.

THEOREM 2.1. The terminating continued fraction

$$\frac{1}{1} + \frac{\lambda_1}{1} + \frac{\lambda_2}{1} + \cdots + \frac{\lambda_n}{1}, \quad \lambda_j \neq 0, \quad j = 1, 2, \ldots, n, \tag{2.3}$$

is in B if and only if there exists real numbers $\{g_j\}_{j=0}^{n}$ such that $0 < g_j < 1$ for $j = 0, 1, \ldots, n-1$, $0 < g_n \leq 1$, and

$$|\lambda_j| \leq g_j(1-g_{j-1}), \quad j = 1, 2, \ldots, n. \tag{2.4}$$

A non-terminating continued fraction (2.1) is in B if and only if there exist real numbers $\{g_j\}_{j=0}^{\infty}$ such that $0 < g_n < 1$ and (2.4) holds for all positive integers n.

Proof. Suppose (2.3) is in B. There exists an $R \geq 1$ such that (2.2) is in the disk $|w| \leq R$ whenever $|\lambda'_j| \leq |\lambda_j|$, $j = 1, 2, \ldots, n$. For each such sequence $\lambda' = \{\lambda'_j\}$, define $w_n = 1$ and

$$(2.5) \qquad w_{j-1} = \frac{1}{1 + \lambda'_j w_j}, \qquad j = 1, 2, \ldots, n .$$

Put $g_0 = 1/R$. Then $0 < g_0 \leq 1$ and $|w_0| \leq 1/g_0$. Assume that for all admissable sequences λ' and for a fixed positive integer $j \leq n$, there exists a g_{j-1}, $0 < g_{j-1} < 1$, such that the values of w_{j-1} in (2.5) are in the disk $|w| < 1/g_{j-1}$. By (2.5) and Lemma 2.1, the values of w_j are always in $|w_j| \leq (1 - g_{j-1})/|\lambda_j|$. Since $w_j = 1$ (when $\lambda'_{j+1} = 0$) must be in this disk, $g_{j-1} < 1$ and $|\lambda_j| \leq 1 - g_{j-1}$. Set $g_j = |\lambda_j|/(1 - g_{j-1})$. Then $0 < g_j \leq 1$. The necessity part of the first statement in the theorem now follows by induction.

Conversely, let the partial numerators of (2.3) obey (2.4) and let $\lambda' = \{\lambda'_j\}_{j=1}^{n}$ be an admissable sequence. Since $|\lambda'_j| \leq |\lambda_j|$ for $j = 1, 2, \ldots, n$, the assumption $|w_j| \leq 1/g_j$ for $j \leq n$ implies by (2.4) and (2.5) that

$$|w_{j-1}| \le \frac{1}{1-|\lambda'_j|\,|w_j|} \le \frac{1}{1-|\lambda_j|\,|w_j|} \le \frac{1}{g_{j-1}} \quad .$$

Also $|w_n| = 1 \le 1/g_n$. By a finite induction, $|w_0| \le 1/g_0$ and the continued fraction (2.3) is in B .

The non-terminating case is an immediate consequence of the previous results and the definition of the class B .

The sufficiency parts of Theorem 2.1 follow also from a result of Scott and Wall [10; 15, p. 45]. Because of the simplicity of the proof, it was included here.

The condition that the continued fraction (2.1) be in the class B implies that (2.2) is in the disk

$$(2.6) \qquad \left|w - \frac{1}{g_0(2-g_0)}\right| \le \frac{1-g_0}{g_0(2-g_0)} \quad .$$

Indeed, by (2.5) and Theorem 2.1

$$\left|\frac{1}{w_{j-1}} - 1\right| \le |\lambda'_j|\,|w_j| \le 1-g_{j-1}$$

which is equivalent to

$$\left| w_{j-1} - \frac{1}{g_{j-1}(2-g_{j-1})} \right| \leq \frac{1-g_{j-1}}{g_{j-1}(2-g_{j-1})} \quad , \quad j = 1, 2, \ldots \quad .$$

Theorem 2.1 and the classical convergence criterion of Pringsheim [9; 15, p. 45] establish the convergence of a continued fraction (2.1) in the class B. A stronger result in this direction is the following one due to Lane and Wall [4].

THEOREM 2.2. _The continued fraction_ (2.1) _is in_ B _if and only if the continued fraction_

$$\frac{1}{1} + \frac{\lambda'_1}{1} + \frac{\lambda'_2}{1} + \cdots + \frac{\lambda'_n}{1} + \cdots$$

converges whenever $|\lambda'_n| \leq |\lambda_n|$, $n = 1, 2, \ldots$. _In this case the convergence is always absolute convergence._

It is useful for some of the applications to have properties of a class of continued fractions which is allied to the class B. In order to define this class, let $\lambda = \{\lambda_n\}_{n=1}^{\infty}$ be a sequence of complex numbers such that $\lambda_n = 0$ implies $\lambda_{n+p} = 0$ for $p = 1, 2, \ldots$. Then the continued fraction

$$(2.7) \qquad 1 + \frac{\lambda_1}{1} + \frac{\lambda_2}{1} + \cdots + \frac{\lambda_n}{1} + \cdots$$

is said to be in the class P when for $n = 1, 2, \ldots$ the values of

$$1 + \frac{\lambda'_1}{1} + \frac{\lambda'_2}{1} + \cdots + \frac{\lambda'_n}{1}$$

are in the closed right-half plane $\operatorname{Re} w \geq 0$ whenever $|\lambda_j'| \leq |\lambda_j|$ for $j = 1, 2, \ldots$. The relation of P to the class B is deduced from the following result.

LEMMA 2.2. _Let_ $A > 0$. _For all_ λ _in the disk_ $|\lambda| \leq A$, $\operatorname{Re} u \geq 0$, _where_ $u = 1 - \lambda v$, _if and only if_ $|v| \leq 1/A$. _In this case_ $|u - 1| \leq 1$.

The proof is similar to that of Lemma 2.1 and is omitted.

In conjunction with the definition of the class B, this shows that (2.6) is in P if and only if the continued fraction

$$\frac{1}{1} + \frac{\lambda_2}{1} + \frac{\lambda_3}{1} + \cdots + \frac{\lambda_n}{1} + \cdots$$

is in B with an $R \geq 1$ such that $|\lambda_1| \leq 1/R$. This and Theorem 1.1 prove

THEOREM 2.3. _A continued fraction_

$$1 + \frac{\lambda_1}{1} + \frac{\lambda_2}{1} + \cdots + \frac{\lambda_n}{1}, \quad \lambda_j \neq 0, \quad j = 1, 2, \ldots, n,$$

is in the class P _if and only if there exist numbers_ $\{g_j\}_{j=1}^n$ _such that_ $0 < g_j < 1$, $j = 1, 2, \ldots, n-1$, $0 < g_n \leq 1$, _and_

$$(2.8) \qquad |\lambda_1| \leq g_1, \quad |\lambda_j| \leq g_j(1 - g_{j-1}), \quad j = 2, 3, \ldots, n.$$

The non-terminating continued fraction (2.6) is in P if and only if there exist numbers $\{g_j\}_{j=1}^{\infty}$ such that $0 < g_n < 1$ and (2.7) holds for all positive integers n.

It is evident from Theorem 2.2 and Theorem 2.3 that (2.6) is absolutely convergent when it is in the class P.

3. Chain sequences and univalence. From results of Leighton and Scott [5] there is a unique, one-to-one correspondence between formal power series $z + \Sigma_{n=2}^{\infty} c_n z^n$ and C-fractions

$$(3.1) \qquad F(z) = \frac{z}{1} - \frac{a_1 z^{\alpha_1}}{1} - \frac{a_2 z^{\alpha_2}}{1} - \cdots - \frac{a_n z^{\alpha_n}}{1} - \cdots,$$

where α_n is a positive integer and $a_{p+n} = 0$ whenever $a_p = 0$ for $n = 1, 2, \ldots$. It is convenient to take $\alpha_{p+n} = 1$ for $n = 0, 1, 2, \ldots$ when $a_p = 0$. For a fixed C-fraction (3.1), let K_F denote the class of formal power series which correspond to continued fractions of the form

$$(3.2) \qquad \frac{z}{1} - \frac{a_1' z^{\alpha_1}}{1} - \frac{a_2' z^{\alpha_2}}{1} - \cdots - \frac{a_n' z^{\alpha_n}}{1} - \cdots,$$

where $|a_n'| \leq |a_n|$, $n = 1, 2, \ldots$. By Theorem 2.2, a necessary and sufficient condition for every $f(z) \in K_F$ to be analytic in $|z| < 1$ is that the continued fraction $F(1)$, obtained from (3.1), be in the class B. In general the radius of univalence of K_F is not one. However, we have the following result in this direction.

THEOREM 3.1 . Each $f(z) \in K_F$ is analytic and univalent in $|z| < 1$ if $\{|a_j|\}_{j=1}^{\infty}$ is a chain sequence with maximal parameters M_j, $j = 0, 1, 2, \ldots$, where $M_0 \neq 0$ and, in case (3.1) terminates with n^{th} partial quotient, $M_{n+p} = 1$ for $p = 0, 1, 2, \ldots$, and

$$(3.3) \qquad \Lambda = \sum_{j=1}^{\infty} |u_j - v_j| \prod_{p=1}^{2j-1} \frac{1-M_{p-1}}{M_p} + \sum_{j=1}^{\infty} |u_j - v_{j+1}| \prod_{p=1}^{2j} \frac{1-M_{p-1}}{M_p} \leq 1 ,$$

where

$$(3.4) \qquad u_j = \sum_{p=1}^{j} \alpha_{2p-1}, \quad v_j = \sum_{p=0}^{j-1} \alpha_{2p}, \quad \alpha_0 = 1, \quad j = 1, 2, \ldots .$$

Furthermore, if $v_j \leq u_j \leq v_{j+1}$ for $j = 1, 2, \ldots$, there is a function in K_F which is not univalent in $|z| < R$ for any $R > 1$.

Proof. Let $f(z) \in K_F$ correspond to the C-fraction (3.2) and let

$$(3.5) \qquad f_{j-1}(z) = \frac{z}{1} - \frac{a_j' z^{\alpha_j}}{1} - \frac{a_{j+1}' z^{\alpha_{j+1}}}{1} - \cdots, \quad j = 1, 2, \ldots .$$

Since $\{|a_j|\}$ is a chain sequence with 0^{th} maximal parameter $M_0 \neq 0$ and since $|a_j' z^{\alpha_j}| \leq |a_j|$ for $j = 1, 2, \ldots$ and $|z| \leq 1$, it follows from Theorems 2.1 and 2.2 that $f(z) \equiv f_0(z)$ and each of the functions $f_{j-1}(z)$ for $j = 2, 3, \ldots$ are analytic in $|z| < 1$. From the results of Section 2, moreover,

$$(3.5) \qquad \left|\frac{f_{j-1}(z)}{z}\right| \leq \frac{1}{M_{j-1}}, \quad j = 1, 2, \ldots, \quad |z| \leq 1 .$$

From (3.4), $v_j < v_{j+1}$ for each j so that $u_j - v_j \neq 0$ or $v_{j+1} - u_j \neq 0$. Thus the hypothesis (3.3) implies there is a sequence of positive integers such that $\prod_{p=1}^{n} (1 - M_{p-1})/M_p$ tends to zero as n tends to ∞ through this sequence. For brevity only the case where this sequence contains an infinitude of even integers $\{2m_k\}$ is treated. The other case is similar.

For any two non zero points, z_1 and z_2, in $|z| \leq 1$, and each positive integer m, it is found from (3.5) that

$$(3.7) \quad z_1 z_2 \left[\frac{1}{f_0(z_2)} - \frac{1}{f_0(z_1)}\right] = z_1 - z_2 - \sum_{j=1}^{m} (z_1^{u_j} z_2^{v_j} - z_1^{v_j} z_2^{u_j}) \prod_{p=1}^{2j-1} a_p' \frac{f_p(z_1) f_p(z_2)}{z_1 z_2}$$

$$- \sum_{j=1}^{m} (z_1^{v_{j+1}} z_2^{u_j} - z_1^{u_j} z_2^{v_{j+1}}) \prod_{p=1}^{2j} a_p' \frac{f_p(z_1) f_p(z_2)}{z_1 z_2} - R_{2m+1}$$

where u_j and v_j are defined by (3.4) and

$$(3.8)\quad R_{2m+1} = a'_{2m+1}[z_1^{u_{m+1}-1} z_2^{v_{m+1}} f_{2m+1}(z_1) - z_1^{v_{m+1}} z_2^{u_{m+1}-1} f_{2m+1}(z_2)] \prod_{p=1}^{2m} a'_p \frac{f_p(z_1)f_p(z_2)}{z_1 z_2}.$$

Since for any pair of positive integers s and t, $|z_1^s z_2^t - z_1^t z_2^s| \leq |s-t||z_1 - z_2|$ when $|z_1| \leq 1$, $|z_2| \leq 1$, it follows from (3.6), (3.7) and the hypothesis $|a'_j| \leq M_j(1 - M_j - 1)$, $j = 1, 2, \ldots,$ that for $z_1 \neq z_2$

$$\left| \frac{z_1 z_2}{z_1 - z_2} \frac{f_0(z_1) - f_0(z_2)}{f_0(z_2) f_0(z_1)} - 1 \right| \leq \sum_{j=1}^{m} |u_j - v_j| \prod_{p=1}^{2j-1} \frac{1 - M_{p-1}}{M_p}$$

$$+ \sum_{j=1}^{m} |v_{j+1} - u_j| \prod_{p=1}^{2j} \frac{1 - M_{p-1}}{M_p} + |R_{2m+1}|,$$

and by (3.8)

$$|R_{2m+1}| \leq 2 \prod_{p=1}^{2m} \frac{1 - M_{p-1}}{M_p}.$$

Let $m = m_k$ so that the right-hand side of the last inequality tends to zero as $m_k \to \infty$. By (3.3), therefore,

$$(3.9)\qquad \left|\frac{z_1 z_2}{z_1 - z_2}\,\frac{f_0(z_1) - f_0(z_2)}{f_0(z_2) f_0(z_1)} - 1\right| \le \Lambda \le 1 .$$

The maximum modulus principle when applied to the limit of the expression (3.7) as $m = m_k \to \infty$, where z_1 is treated as an independent variable, shows that strict inequality holds in (3.9) when $|z_1| < 1$, $|z_2| < 1$. In particular, $f_0(z_1) \neq f_0(z_2)$ for $z_1 \neq z_2$ in $|z| < 1$. This proves the first part of the theorem.

Let $m = m_k \to \infty$ in (3.7) and then divide by $z_1 - z_2$. If $z_1 \to z_2 = z$ for $|z_1| \le 1$, $|z_2| \le 1$, we obtain

$$\frac{z^2 f_0'(z)}{[f_0(z)]^2} = 1 - \sum_{j=1}^{\infty} (u_j - v_j) z^{u_j + v_j - 1} \prod_{p=1}^{2j-1} a_p' \left[\frac{f_p(z)}{z}\right]^2$$

$$- \sum_{j=1}^{\infty} (v_{j+1} - u_j) z^{u_j + v_{j+1} - 1} \prod_{p=1}^{2j} a_p' \left[\frac{f_p(z)}{z}\right]^2 .$$

Suppose now that $v_j \le u_j \le v_{j+1}$ for $j = 1, 2, \ldots$ and that

$$f_0(z) = \frac{z}{1} - \frac{|a_1|z^{\alpha_1}}{1} - \frac{|a_2|z^{\alpha_2}}{1} - \cdots - \frac{|a_n|z^{\alpha_n}}{1} - \cdots, \tag{3.11}$$

which is clearly in K_F. From the definition of a maximal parameter sequence of a chain sequence [15, p. 81], the functions (3.5) for this C-fraction satisfy

$$f_{j-1}(1) = \frac{1}{1} - \frac{|a_j|}{1} - \frac{|a_{j+1}|}{1} - \cdots = \frac{1}{M_{j-1}}, \quad j = 1, 2, \ldots .$$

Since $|a_j| = M_j(1 - M_{j-1})$, $j = 1, 2, \ldots$, (3.3) and (3.10) show that

$$\frac{f_0'(1)}{[f_0(1)]^2} = 1 - \Lambda$$

for the function $f_0(z)$ of (3.11). Therefore if $\Lambda = 1$, $f_0(z)$ is not univalent in any disk $|z| < R$ for $R > 1$. This completes the proof.

This theorem extends results of Thale [13], Perron [8], Scott and Merkes [7]. For results equivalent to those previously obtained, put $\alpha_j = \alpha$ and $M_j = g$, $0 < g < 1$, for $j = 1, 2, \ldots$ in Theorem 3.1.

If a formal power series $F(z)$ corresponds to a J-fraction, it is often possible to obtain a larger set of univalence of $F(z)$ than that of Theorem 3.1. A result of this kind is given by the next theorem. For simplicity, the theorem is stated in terms of the reciprocal variable $\zeta = 1/z$.

THEOREM 3.2. *Let* b_n *and* $\beta_n > 0$ *be complex numbers for* $n = 1, 2, \ldots$. *Let* D *be the set of all* ζ *such that* $|\zeta - b_n| > \beta_n$ *for all positive integers* n. *The* J-*fraction*

$$(3.12) \qquad \frac{1}{\zeta - b_1} - \frac{a_1^2}{\zeta - b_2} - \frac{a_2^2}{\zeta - b_3} - \cdots - \frac{a_n^2}{\zeta - b_n} - \cdots$$

represents an analytic univalent function in D *provided*

$$\{|a_n|^2/\beta_n \beta_{n+1}\}_{n=1}^{\infty}$$

is a chain sequence with maximal parameter sequence

$$\{M_n\}_{n=0}^{\infty}, \quad M_0 \neq 0,$$

such that

$$\sum_{n=1}^{\infty} \frac{\beta_1}{\beta_n} \prod_{p=1}^{n} \frac{1 - M_{p-1}}{M_p} < 1.$$

The proof is similar to that of Theorem 3.1 and is, therefore, omitted.

In particular, when $|a_n| \leq N/3$, $\beta_n \geq \beta_1 > 0$, $n = 1, 2, \ldots$, then (3.12) is analytic and univalent in the common part of the regions

$|\zeta - b_n| > \sqrt{2}\, N/2$. This statement includes Thale's results on the J-fraction [6, 13]. Indeed, if $|b_n| \leq M/3$, $n = 1, 2, \ldots$, then the domain of univalence (3.12) is $|\zeta| > (3\sqrt{2}\, N + 2M)/6$. Moreover, if $\operatorname{Im} b_n \leq 0$, $n = 1, 2, \ldots$, which is the case whenever (3.12) is positive definite, then $\operatorname{Im} \zeta > \sqrt{2}\, N/2$ is a domain of univalence of (3.12). Each of these can be shown to be sharp. For, by a suitable choice of the real number a, the function

$$\frac{2}{3(\zeta - a) - \sqrt{(\zeta - a)^2 + 4N^2/9}} = \frac{1}{\zeta - a} - \frac{N^2/9}{\zeta - a} + \frac{N^2/9}{\zeta - a} + \frac{N^2/9}{\zeta - a} + \cdots$$

whose derivative vanishes at $\zeta = a + \sqrt{2}\, Ni/2$, is not univalent in a given open region which properly contains one of these domains of univalence.

4. Starlikeness and chain sequences. A lower bound for the radius of starlikeness of the class K_F, associated with the C-fraction (3.1), is given by the following theorem.

THEOREM 4.1. *Each $f(z) \in K_F$ is analytic, univalent, and starlike in $|z| < 1$ provided $\{|a_j|\}_{j=1}^{\infty}$ is a chain sequence with maximal parameters $\{M_j\}_{j=0}^{\infty}$, where $M_0 \neq 0$, and*

$$(4.1) \qquad \Lambda \leq \sqrt{M_0(2-M_0)} \ ,$$

where Λ is defined in (3.3).

Proof. Let $f(z) \in K_F$ correspond to the C-fraction (3.2). By (3.9), where $z_1 \to z_2 = z$,

$$\left| \frac{zf'(z)}{f(z)} - \frac{f(z)}{z} \right| \leq \Lambda \left|\frac{f(z)}{z}\right|, \quad |z| \leq 1 .$$

This implies $\text{Re}\{zf'(z)/f(z)\} > 0$ for $|z| < 1$, which is a well-known characterization of starlikeness for normalized univalent functions in the unit disk, provided

$$(4.2) \qquad \cos\theta \geq \Lambda, \quad \theta = \arg(f(z)/z) .$$

Since by (2.6)

$$\left| \frac{f(z)}{z} - \frac{1}{M_0(2-M_0)} \right| \leq \frac{1-M_0}{M_0(2-M_0)} , \quad |z| \leq 1 ,$$

it follows that

$$\cos\theta \leq \sqrt{M_0(2-M_0)} ,$$

which in conjunction with (4.2) proves the sufficiency of (4.1) for starlikeness.

In particular, when $M_n = M_0$, $\alpha_n = 1$, $n = 1, 2, \ldots$, $M_0 > .60$ which improves the lower bound for M_0 of $(3-\sqrt{3})/2$ previously obtained [7]. It is conjectured by the authors that $M_0 \geq 2-\sqrt{2}$, the lower bound of M_0 for univalence in this case.

5. Some lemmas from the problem of moments. Before a second application of chain sequences to univalence is discussed, it is helpful to have at hand some elementary consequences of the Stieltjes and the Hausdorff moment problems. For this purpose, let $\{k_n\}_{n=1}^{\infty}$ be a sequence of positive numbers and let $F(z)$ denote the formal power series which corresponds to the S-fraction

$$(5.1) \qquad \frac{1}{1} + \frac{k_1 z}{1} + \frac{k_2 z}{1} + \cdots + \frac{k_n z}{1} + \cdots .$$

From results of Stieltjes, there exists a bounded non-decreasing function $\alpha(t)$ on $0 \leq t < \infty$ such that

$$(5.2) \qquad F(z) \sim I(z;\alpha) = \int_0^{\infty} \frac{d\alpha(t)}{1+zt} .$$

The function $I(z;\alpha)$ is analytic for z in the complex plane cut along the negative real axis [17, p. 328].

LEMMA 3.1. Let the sequence $\{k_n r\}_{n=1}^{\infty}$ be a chain sequence if and only if $0 < r \leq 1$. Then the formal power series $F(z)$ corresponding to (5.1) converges in the disk $|z| < 1$ and represents a function which is analytic in the complex plane cut from $-\infty$ to -1 along the negative real axis and which has a singularity at $z = -1$.

Proof. Since $\{k_n\}$ is a chain sequence, the S-fraction (5.1) converges and represents an analytic function in the z-plane cut from $-\infty$ to -1 along the negative real axis [15, p. 46]. Hence the power series $F(z)$ is convergent for $|z| < 1$ and there is a bounded non-decreasing function $\alpha(t)$ on $0 \leq t \leq 1$ such that

$$F(z) = \int_0^1 \frac{d\alpha(t)}{1+zt}$$

[15, p. 263]. Suppose now $\alpha(t)$ has no point of increase at $t = 1$, i.e., there is an ϵ, $0 < \epsilon < 1$, such that $\alpha(t)$ is constant on $1-\epsilon < t \leq 1$. This implies

$$F(z) = \int_0^{1-\epsilon} \frac{d\alpha(t)}{1+zt} = \int_0^1 \frac{d\alpha[(1-\epsilon)t]}{1+\zeta t}$$

$$= \frac{1}{1} + \frac{k_1\zeta/(1-\epsilon)}{1} + \frac{k_2\zeta/(1-\epsilon)}{1} + \cdots,$$

where $\zeta = (1-\epsilon)z$. Results on the Hausdorff moment problem [15, p. 263] and the last integral representation now imply that $\{k_n/(1-\epsilon)\}_{n=1}^{\infty}$ is a chain sequence. This is contrary to the hypothesis that $\{k_n r\}_{n=1}^{\infty}$ is not a chain sequence for $r > 1$. Hence $\alpha(t)$ has a point of increase at $t = 1$.

Define

$$\beta(s) = \int_{1/(1+s)}^{1} \frac{d\alpha(t)}{t}, \quad 0 \leq s < \infty .$$

Clearly $\beta(s)$ is non-decreasing and, since $\beta(0) = 0$,

$$\beta(s) \geq \alpha(1) - \alpha\left(\frac{1}{1+s}\right) > 0, \quad s > 0 .$$

This function has a point of increase at $s = 0$. Since

$$\int_0^\infty \frac{d\beta(s)}{s+1+z} = \int_0^1 \frac{d\alpha(t)}{1+zt} = F(z),$$

it follows from well-known results on Stieltjes transforms [17, p. 337] that $F(z)$ has a singularity at $z = -1$.

LEMMA 3.2. Suppose that for each $r > 0$, the sequence $\{k_n r\}_{n=1}^{\infty}$ is not a chain sequence. Then the power series $F(z)$ corresponding to (5.1) diverges in each neighborhood of zero.

Proof. Suppose that the series

$$F(z) = \sum_{n=0}^{\infty} (-1)^n a_n z^n \sim \int_0^\infty \frac{d\alpha(t)}{1+zt},$$

where

$$a_n = \int_0^\infty t^n \, d\alpha(t) , \quad n = 0, 1, 2, \dots ,$$

is convergent for $|z| \leq R$, $R > 0$. Then $a_n R^n \to 0$ as $n \to \infty$. Suppose $\alpha(t)$ has a point of increase at ∞. Then for $T > 1/R$,

$$a_n R^n \geq \int_T^\infty (Rt)^n \, d\alpha(t) \geq (RT)^n [\alpha(\infty) - \alpha(T)] > 0 .$$

Thus $a_n R^n \to \infty$ as $n \to \infty$. This contradiction shows there is a $T > 0$ such that $\alpha(t)$ is constant for $T < t \leq \infty$. Therefore,

$$\int_0^\infty \frac{d\alpha(t)}{1+zt} = \int_0^T \frac{d\alpha(t)}{1+zt} = \int_0^1 \frac{d\alpha(Ts)}{1+\zeta s}$$

$$\sim \frac{1}{1} + \frac{k_1 \zeta/T}{1} + \frac{k_2 \zeta/T}{1} + \cdots ,$$

where $\zeta = Tz$. From results on the Hausdorff moment problem, the latter implies $\{k_n/T\}_{n=1}^\infty$ is a chain sequence, which is contrary to the hypothesis. Hence the series $F(z)$ is divergent for $z \neq 0$.

If (5.1) terminates with $n^{\underline{th}}$ partial quotient, then this continued fraction represents a rational function whose poles are negative real, simple, and have positive residue. Therefore it is found that Lemma 3.1 remains valid when the sequence $\{k_p\}_{p=1}^\infty$ is such that $k_p > 0$ for $p = 1, 2, \dots, n-1$;

$k_p = 0$ for $p = n, n+1, \ldots$. Moreover, for each such sequence, $\{k_p r\}_{p=0}^{\infty}$ is a chain sequence for some $r > 0$. Hence the hypothesis of Lemma 3.2 is not fulfilled in this case.

6. The radii of univalence and of starlikeness of the class Π_f .

Let

$$(6.1)\qquad f(z) = \sum_{n=1}^{\infty} c_n z^n , \quad zf'(z) = \sum_{n=1}^{\infty} n c_n z^n , \quad c_1 \neq 0 ,$$

be formal power series. From the one-to-one correspondence between formal power series and C-fraction [5] ,

$$(6.2)\qquad \frac{zf'(z)}{f(z)} \sim 1 - \frac{a_1 z^{\alpha_1}}{1} - \frac{a_2 z^{\alpha_2}}{1} - \cdots - \frac{a_n z^{\alpha_n}}{1} - \cdots ,$$

where $\{\alpha_n\}$ and $\{a_n\}$ are respectively sequences of positive integers and of complex numbers, and the expression on the left is the formal quotient of the series (6.1). The continued fraction (6.2) terminates with $k \underline{\text{th}}$ partial quotient if $a_j \neq 0$ for $j = 1, 2, \ldots, k$ and $a_{k+1} = 0$.

For a fixed series $f(z)$ as in (6.1), let Π_f denote the class of formal power series $g(z) = \sum_{n=1}^{\infty} c_n^* z^n$, $c_1^* \neq 0$, such that

$$(6.3)\qquad \frac{zg'(z)}{g(z)} \sim 1 - \frac{a_1^* z^{\alpha_1}}{1} - \frac{a_2^* z^{\alpha_2}}{1} - \cdots - \frac{a_n^* z^{\alpha_n}}{1} - \cdots ,$$

where $|a_n^*| \le |a_n|$, $n = 1, 2, \ldots$, and the sequences $\{\alpha_n\}$, $\{a_n\}$ are given in the correspondence (6.2). Let $U(\Pi_f)$ denote the radius of univalence of the class Π_f, i.e., $U(\Pi_f)$ is the suprema of the $r \ge 0$ for which each member of Π_f is an analytic univalent function in $|z| < r$. It is agreed to put $U(\Pi_f) = 0$ in case there is a member of Π_f which is not analytic at $z = 0$. The radius of starlikeness with respect to the origin $S(\Pi_f)$ is defined in a similar manner. Evidently, $U(\Pi_f) \ge S(\Pi_f) \ge 0$. Moreover, if $g \in \Pi_f$, then $U(\Pi_g) \ge U(\Pi_f)$ and $S(\Pi_g) \ge S(\Pi_f)$.

THEOREM 6.1. For a fixed power series $f(z)$ in (6.1), the correspondence (6.2) holds. Let r_0 be the suprema of the $r \ge 0$ for which $\{|a_n| r^{\alpha_n}\}_{n=1}^{\infty}$ is a chain sequence. Then $r_0 \le S(\Pi_f) \le U(\Pi_f)$. Moreover, if the sequence $\{|a_n| r_0^{\alpha_n}\}_{n=1}^{\infty}$ is a chain sequence with uniquely determined parameters, then $S(\Pi_f) = U(\Pi_f) = r_0$.

Proof. First, it is evident from results on chain sequences [15, p. 86] that $\{|a_n| r_0^{\alpha_n}\}_{n=1}^{\infty}$ is itself a chain sequence. Now if $r_0 = 0$, there is nothing to prove. If $r_0 > 0$, for each $g(z) \in \Pi_f$, the C-fraction expansion (6.3) converges in the disk $|z| < r_0$ by Theorem 2.2. It follows that the power series $zg'(z)/g(z)$ converges in this disk [5] and, hence, that $g(z)$ is analytic in $|z| < r_0$. Furthermore, when $|z| \le r_0$ the continued fraction (6.3) is in the class P of §2 and, therefore,

$$\operatorname{Re} \frac{zg'(z)}{g(z)} \geq 0, \quad |z| \leq r_0 .$$

Since $g(0) = 0$, $g'(0) \neq 0$, this implies that $g(z)$ is univalent and starlike with respect to the origin [11] for $|z| < r_0$. Thus $r_0 \leq S(\Pi_f) \leq U(\Pi_f)$.

Let $f_0(z)$ denote the formal series for which

$$\frac{zf_0'(z)}{f_0(z)} \sim 1 - \frac{|a_1|z^{\alpha_1}}{1} - \frac{|a_2|z^{\alpha_2}}{1} - \cdots - \frac{|a_n|z^{\alpha_n}}{1} - \cdots .$$

Then $f_0(z) \in \Pi_f$. If $\{M_j\}_{j=0}^{\infty}$ denotes the maximal parameter sequence of the chain sequence $\{|a_n| r_0^{\alpha_n}\}_{n=1}^{\infty}$, it is known [15, p. 81] that $M_0 = r_0 f'(r_0)/f(r_0)$. Since $M_0 = 0$ when the parameters are uniquely determined [15, p. 82], $f_0'(z)$ has a zero or $f_0(z)$ has a singularity at $z = r_0$. In either case, the function $f_0(z)$ is not analytic and univalent in any disk $|z| < R$ for $R > r_0$. This proves the last statement of the theorem.

THEOREM 6.2 . _Let_ $f(z)$ _be a power series_ (6.1) _and let_

$$\frac{zf'(z)}{f(z)} \sim 1 - \frac{a_1 z^{\alpha}}{1} - \frac{a_2 z^{\alpha}}{1} - \cdots - \frac{a_n z^{\alpha}}{1} - \cdots , \tag{6.4}$$

where α *is a positive integer. Then* $U(\Pi_f) = S(\Pi_f) = r_0$, *where* r_0 *is the suprema of the* $r \geq 0$ *such that* $\{|a_n| r^\alpha\}$ *is a chain sequence. If* $f_0(z)$ *is a function such that*

$$(6.5) \qquad \frac{zf_0'(z)}{f_0(z)} \sim 1 - \frac{|a_1|z^\alpha}{1} - \frac{|a_2|z^\alpha}{1} - \cdots - \frac{|a_n|z^\alpha}{1} - \cdots,$$

then r_0 *is the smallest non-negative zero or singularity of* $f_0'(z)$.

Proof. By Theorem 6.1, $r_0 \leq S(\Pi_f) \leq U(\Pi_f)$. If $r_0 > 0$, then by Lemma 5.1 the ratio $f_0(z)/zf_0'(z)$ obtained from (6.5) is analytic in $|z| < r_0$ and has a singularity at $z = r_0$. Thus $f_0'(z)$ is analytic and non-zero in $|z| < r_0$ and has a zero or a singularity at $z = r_0$. In any case $f_0(z)$ is not analytic and univalent in $|z| < R$ for any $R > r_0$. Therefore, $r_0 = U(\Pi_f) = S(\Pi_f)$. On the other hand, if $r_0 = 0$, the function $f_0(z)$ is not analytic at $z = 0$ by Lemma 5.2. Hence $U(\Pi_f) = S(\Pi_f) = 0$ in this case and the proof is complete.

In some special cases it is easy to obtain an upper bound for the radius of starlikeness of the function $f(z)$ itself from the expansion (6.4). One such result is given by the next theorem.

THEOREM 6.3. *Let* $f(z)$ *be a series* (6.1) *such that* (6.4) *holds.*

If $M = \sup\limits_{n \geq 2} |a_n| \leq (3 - 2\sqrt{2})\,|a_1|$, **then** $f(z)$ **is not starlike in**

$|z| < R$ **for** $R > R_0$, **where**

$$R_0^\alpha = \frac{3|a_1| - M - \sqrt{|a_1|^2 - 6|a_1|M + M^2}}{2|a_1|^2} \quad . \tag{6.6}$$

Proof. If $|z| \leq R_0$, $|a_n z^\alpha| \leq MR_0^\alpha < 1/4$, $n = 2, 3, \ldots$, by (6.6). Set $g(1-g) = MR_0^\alpha$ and

$$\frac{1}{2} \leq g = \frac{1 + \sqrt{1 - 4MR_0^\alpha}}{2} \leq 1 \ . \tag{6.7}$$

Since this implies the continued fraction

$$w = \frac{1}{1} + \frac{a_2 z^\alpha}{1} + \frac{a_3 z^\alpha}{1} + \cdots + \frac{a_n z^\alpha}{1} + \cdots$$

is in the class B of §2 for $z = R_0$, it converges for $|z| \leq R_0$ by Theorem 1.2 and $\operatorname{Re} w \geq 1/(2-g)$ by (2.6). Now when $z = R_0 \exp(-\theta/\alpha)$, where $\theta = \arg a_1$,

$$\operatorname{Re}\frac{zf'(z)}{f(z)} = 1 - |a_1|R_0^\alpha \operatorname{Re} w \leq 1 - \frac{|a_1|R_0^\alpha}{2-g} \leq 0$$

by (6.4), (6.6), and (6.7). Consequently, $f(z)$ is not starlike in any disk concentric to $|z| < R_0$ with larger radius.

7. Univalence of Bessel functions. A study of the univalence of the function $F_\nu(z) = z^{1-\nu} J_\nu(z)$, where $J_\nu(z)$ is a Bessel function of order ν, was recently initiated by Kreyszig and Todd [3] for $\nu > -1$ and by Brown [1, 2] for some complex values of ν. Wilf [18] has simplified the proof of the main result in [3] and has replaced some of the inequalities for the radius of univalence of $F_\nu(z)$ with large ν by asymptotic equalities. These results and extensions of them are corollaries of the theorems in the previous section of this paper.

From the recurrence formulas [16, p. 45]

$$zJ_{\nu+1}(z) = 2\nu J_\nu(z) - z J_{\nu-1}(z) \quad ,$$

$$zJ_{\nu+1}(z) = \nu J_\nu(z) - z J'_\nu(z) \quad ,$$

it follows that for $\nu \neq -1, -2, \ldots$

$$\frac{zF'_\nu(z)}{F_\nu(z)} \sim 1 - \frac{\frac{1}{2} z^2/(\nu+1)}{1} - \frac{\frac{1}{4} z^2/(\nu+1)(\nu+2)}{1} - \cdots , \tag{7.1}$$

where $F_\nu(z) = z^{1-\nu} J_\nu(z)$. The continued fraction converges throughout the z-plane except at the zeros of $J_\nu(z)$ and, therefore, the correspondence symbol in (7.1) can be replaced by an equality [14; 15, p. 347 ff].

THEOREM 7.1 . Let $x = \mathrm{Re}\,\nu > -1$. The radius of starlikeness ρ_ν of $F_\nu(z) = z^{1-\nu} J_\nu(z)$ is not less than the smallest positive zero of $F_x'(z)$. Moreover,

$$\rho_\nu^2 \geq 2|\nu+1|\{1 - \frac{1}{1 + 2|\nu+2|[1 - 1/2|\nu+3|]}\} \tag{7.2}$$

and

$$\lim_{\nu \to \infty} \frac{\rho_\nu^2}{|\nu|} = 2 \ . \tag{7.3}$$

Proof. Since $|\nu + n| \geq x + n > 0$ for $n = 1, 2, \dots$, $F_\nu(z)$ is in the class Π_{F_x} . In view of the fact that $F_x(z)$ is an entire function, the first part of the theorem is now a consequence of Theorem 6.2 .

Let $|z| \leq r$, where r^2 is the quantity on the right-hand side of the inequality (7.2). Put

$$0 < g_1 = \frac{r^2}{2|\nu+1|} = 1 - \frac{1}{2|\nu+2|[1 - 1/2|\nu+3|] + 1} < 1 \ ,$$

$$g_{n+1} = \frac{r^2}{4|\nu+n|\,|\nu+n+1|(1-g_n)} \ , \quad n = 1, 2, \dots \ .$$

Since $|\nu + n + 1| > |\nu + n|$ for $n = 1, 2, \ldots$, the assumption $0 < g_{n-1} < 1$, $0 < g_n \leq g_{n-2} < 1$, $n > 2$, implies

$$0 < g_{n+1} \leq \frac{r^2}{4|\nu + n - 1|\,|\nu + n - 2|(1 - g_{n-2})} = g_{n-1} < 1 .$$

Now $g_1 > g_3 = |\nu + 1| g_1 / |\nu + 2|$ and $0 < g_2 = 1 - 1/2|\nu + 3| < 1$.

It follows by induction that $0 < g_n < 1$ for $n = 1, 2, \ldots$ and, therefore, that the sequence $r^2/2|\nu + 1|$, $r^2/4|\nu + 1|\,|\nu + 2|$, ... is a chain sequence. Consequently $r \leq r_0$, where r_0 is defined in Theorem 6.2 . Since $\rho_\nu \geq r_0$, (7.2) is now proved.

Finally, for $|\nu + 2| \geq (3 + 2\sqrt{2})/2$,

$$M = \sup_{n \geq 2} \frac{1}{4|\nu + n|\,|\nu + n + 1|} = \frac{1}{4|\nu + 1|\,|\nu + 2|} \leq \frac{3 - 2\sqrt{2}}{2|\nu + 1|} .$$

By Theorem 6.3 , this implies

$$\frac{\rho_\nu^2}{|\nu + 1|} \leq 3 - \frac{1}{2|\nu + 2|} - \sqrt{1 - \frac{3}{|\nu + 2|} + \frac{1}{4|\nu + 2|^2}} .$$

(7.3) is a consequence of this and (7.2).

For real $\nu > -1$, the bound in (7.2) appears to be a good estimate of ρ_ν. Indeed, it is easy to show by the method used in the proof of Theorem 6.3 that $\operatorname{Re}\{z_0 F'_\nu(z_0)/F_\nu(z_0)\} \leq 0$ when $\nu > -1$ and

$$z_0^2 = 2(\nu+1)\{1 - \frac{1}{2(\nu+2)+1}\} .$$

Theorem 6.2 and (7.1) can be used to obtain information on univalence and starlikeness of $F_\nu(z)$ when $\operatorname{Re}\nu \leq -1$. Moreover, it is possible to obtain from the continued fraction of Gauss [15, p. 347] analogues of the preceeding results for the confluent hypergeometric functions ${}_1F_1(z; a, b)$.

REFERENCES

1. R. K. Brown, Univalence of Bessel functions, Proc. Amer. Math. Soc., 11(1960), 278-283.

2. ________, Univalent solutions of $W'' + pW = 0$, Can. Jour. Math., 14(1962), 69-78.

3. E. O. Kreyszig and J. Todd, The radius of univalence of Bessel functions, Ill. Jour. Math., 4(1960), 143-149.

4. R. E. Lane and H. S. Wall, Continued fractions with absolutely convergent even and odd parts, Trans. Amer. Math. Soc., 67(1949), 366-380.

5. W. Leighton and W. T. Scott, A general continued fraction expansion, Bull. Amer. Math. Soc., 45(1939), 596-605.

6. E. P. Merkes, Bounded J-fractions and univalence, Mich. Math. Jour., 6(1959), 395-400.

7. E. P. Merkes and W. T. Scott, On univalence of a continued fraction, Pac. Jour. Math., 10(1960), 1361-1369.

8. O. Perron, Über ein Schlichtheitsschranke von James S. Thale, S.-B. Nat. Kl. Bayer. Wiss., (1956), 233-236.

9. A. Pringsheim, Über einige konvergenzkreiterien für Kettenbrüche mit komplexem Gliedern, Sb. Munchen, 35(1905).

10. W. T. Scott and H. S. Wall, A convergence theorem for continued fractions, Trans. Amer. Math. Soc., 47(1940), 155-172.

11. L. Špaček, Contribution à la théorie des fonctions univalentes, Časopis pro Pěstování Matematiky a Fysiki, 62(1932), 12-19.

12. T. J. Stieltjes, Recherches sur les fractions continues, Oeavres, 2, 402-566.

13. J. S. Thale, Univalence of continued fractions and Stieltjes transforms, Proc. Amer. Math. Soc., 7(1956), 232-244.

14. E. V. Van Vleck, On the convergence of the continued fraction of Gauss and other continued fractions, Ann. Math., 3(1901), 1-18.

15. H. S. Wall, Analytic theory of continued fractions, Van Nostrand, N. Y., 1948.

16. G. N. Watson, A treatise on the theory of Bessel functions, Macmillan, N. Y., 1944.

17. D. V. Widder, The Laplace transform, Princeton, N. J., 1941.

18. H. S. Wilf, The radius of univalence of certain entire functions, Ill. Jour. Math., 6(1962), 242-244.